Colliding neutrino fields in general relativity

1976 J. Phys. A: Math. Gen. 9 45

(http://iopscience.iop.org/0305-4470/9/1/009)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

Colliding neutrino fields in general relativity

J B Griffiths

Department of Mathematics, University of Technology, Loughborough, Leicestershire LE11 3TU, UK

Received 2 September 1975

Abstract. The neutrino field equations are given in Newman–Penrose formalism, and an exact solution of the Einstein-neutrino equations is obtained which describes the collision and subsequent interaction of two neutrino fields. The gravitational interaction of the two fields is found to be completely different from that between two similar electromagnetic fields.

1. Introduction

An interesting feature of Einstein's general theory of relativity is that, being a non-linear theory, it does not allow a simple superposition of fields. Thus, waves are not able to pass through each other without a significant interaction through the gravitational field equations. In order to gain an understanding of this process, it is important to obtain exact solutions describing the collision of two waves and their subsequent interaction. Szekeres (1972) has obtained an exact solution describing the collision of plane gravitational waves, and has shown that an essential singularity necessarily is induced in the space–time. This may be described as the effect of a mutual focusing and amplification of the two waves (Penrose 1966). Bell and Szekeres (1974) have also given an exact solution describing colliding electromagnetic shock waves. This solution also exhibits a mutual focusing effect, but for the special case described the singularity which appears is non-essential.

The purpose of this work is to derive an exact solution describing a collision of neutrino fields. It is shown that the interaction occuring between neutrino fields is of a different nature to that occuring between electromagnetic or gravitational fields.

In order to compare the interaction of neutrinos with that of gravitational and electromagnetic waves, it is convenient to closely follow the methods of Szekeres (1972) and Bell and Szekeres (1974). It is therefore necessary initially to express the neutrino equations in the Newman–Penrose formalism. The notation of this paper will follow as closely as possible that of Newman and Penrose (1962).

2. The neutrino equations

The neutrino field is defined by a two-component spinor ϕ_A which satisfies the neutrino Weyl equation

$$\sigma^{\mu}{}_{A\dot{B}}\phi^{A}{}_{;\mu}=0, \qquad \text{or} \qquad \phi^{A}{}_{;A\dot{B}}=0. \tag{2.1}$$

It must also satisfy Einstein's field equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=-kE_{\mu\nu}$$

where the energy–momentum tensor is given by

$$E_{\mu\nu}=\mathrm{i}[\sigma_{\mu A\dot{B}}(\phi^A\bar{\phi}^{\dot{B}}{}_{;\nu}-\bar{\phi}^{\dot{B}}\phi^A{}_{;\nu})+\sigma_{\nu A\dot{B}}(\phi^A\bar{\phi}^{\dot{B}}{}_{;\mu}-\bar{\phi}^{\dot{B}}\phi^A{}_{;\mu})].$$

In order to express these in Newman–Penrose formalism it is necessary to introduce two basis spinors o_A and ι_A normalized by the condition

$$o_A\iota^A=-\iota_A o^A=1.$$

These give rise to the null tetrad

$$l^\mu=\sigma^\mu{}_{A\dot{B}}o^A\bar{o}^{\dot{B}}, \qquad n^\mu=\sigma^\mu{}_{A\dot{B}}\iota^A\bar{\iota}^{\dot{B}}$$

$$m^\mu=\sigma^\mu{}_{A\dot{B}}o^A\bar{\iota}^{\dot{B}}, \qquad \bar{m}^\mu=\sigma^\mu{}_{A\dot{B}}\iota^A\bar{o}^{\dot{B}}.$$

The neutrino spinor may now be written in component form as

$$\phi_A=\phi o_A+\psi\iota_A.$$

In previous work on neutrino fields (eg Griffiths and Newing 1971, Wainwright 1971) it was found convenient to align one basis spinor with the neutrino spinor so that $\psi=0$. However, for the problem considered here this simplification is not possible and the neutrino equation (2.1), in terms of the Newman–Penrose spin coefficients and differential operators, takes the form

$$\mathrm{D}\phi+\bar{\delta}\psi=(\rho-\epsilon)\phi+(\alpha-\pi)\psi$$

$$\delta\phi+\Delta\psi=-(\beta-\tau)\phi-(\mu-\gamma)\psi.$$

The components of the Ricci tensor now take the apparently complicated form

$$\Phi_{00}=\mathrm{i}k[\psi\mathrm{D}\bar{\psi}-\bar{\psi}\mathrm{D}\psi+\kappa\phi\bar{\psi}-\bar{\kappa}\psi\bar{\phi}+(\epsilon-\bar{\epsilon})\psi\bar{\psi}]$$

$$\Phi_{01}=\tfrac{1}{2}\mathrm{i}k[\psi\delta\bar{\psi}-\bar{\psi}\delta\psi-\psi\mathrm{D}\bar{\phi}+\bar{\phi}\mathrm{D}\psi-\kappa\phi\bar{\phi}+\sigma\phi\bar{\psi}-(\bar{\rho}+\epsilon+\bar{\epsilon})\psi\bar{\phi}+(\beta-\bar{\alpha}-\bar{\pi})\psi\bar{\psi}]$$

$$\Phi_{02}=-\mathrm{i}k[\psi\delta\bar{\phi}-\bar{\phi}\delta\psi+\sigma\phi\bar{\phi}+(\bar{\alpha}+\beta)\psi\bar{\phi}+\lambda\psi\bar{\psi}]$$

$$\Phi_{11}=\tfrac{1}{2}\mathrm{i}k[\phi\mathrm{D}\bar{\phi}-\bar{\phi}\mathrm{D}\phi+\psi\Delta\bar{\psi}-\bar{\psi}\Delta\psi+(\bar{\epsilon}-\epsilon)\phi\bar{\phi}+(\tau+\bar{\pi})\phi\bar{\psi}-(\bar{\tau}+\pi)\psi\bar{\phi}$$
$$+(\gamma-\bar{\gamma})\psi\bar{\psi}]$$

$$=\tfrac{1}{2}\mathrm{i}k[-\phi\delta\bar{\psi}+\bar{\psi}\delta\phi-\psi\bar{\delta}\bar{\phi}+\bar{\phi}\bar{\delta}\psi+(\bar{\rho}-\rho)\phi\bar{\phi}+(\bar{\alpha}+\beta)\phi\bar{\psi}-(\alpha+\bar{\beta})\psi\bar{\phi}+(\mu-\bar{\mu})\psi\bar{\psi}]$$

$$\Phi_{12}=\tfrac{1}{2}\mathrm{i}k[\phi\delta\bar{\phi}-\bar{\phi}\delta\phi-\psi\Delta\bar{\phi}+\bar{\phi}\Delta\psi+(\bar{\alpha}-\beta-\tau)\phi\bar{\phi}+\bar{\lambda}\phi\bar{\psi}-(\mu+\gamma+\bar{\gamma})\psi\bar{\phi}-\bar{\nu}\psi\bar{\psi}]$$

$$\Phi_{22}=\mathrm{i}k[\phi\Delta\bar{\phi}-\bar{\phi}\Delta\phi+(\bar{\gamma}-\gamma)\phi\bar{\phi}+\bar{\nu}\phi\bar{\psi}-\nu\psi\bar{\phi}]$$

$$\Lambda=0.$$

This now enables neutrino fields to be considered in terms of the Newman–Penrose formalism.

3. Boundary conditions

The situation under consideration is that of two colliding null fields. It is always possible to choose a frame of reference in which the two fields approach from exactly opposite spatial directions, so that only 'head on' collisions need to be considered. It is convenient to choose two null coordinates $x^0 = u$, $x^1 = v$ initially parallel to the two waves, with the wavefronts given by $u = 0$ and $v = 0$ respectively. This situation is described in figure 1.

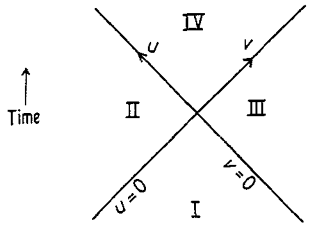

Figure 1. For colliding neutrino fields, region I is taken to be flat, regions II and III contain neutrino fields approaching from opposite directions, and region IV represents the interaction region. Null coordinates u and v are chosen for convenience.

The metric is permitted to take a different form in the four regions indicated in figure 1. However, it must be smoothly joined on the null boundaries $u = 0$ and $v = 0$, the appropriate junction conditions being those of O'Brien and Synge (1952) (see Robson 1973).

In order to determine a solution in the interaction region IV, it is necessary to specify the fields in regions I, II and III. Region I is taken to be flat since no other fields are assumed to be present. A convenient choice of neutrino field in region II is that described by the metric (Griffiths 1972):

$$ds^2 = 2\,du\,dx^1 + h(u)[(x^2)^2 + (x^3)^2]\,du^2 - (dx^2)^2 - (dx^3)^2.$$

In this metric however the coordinate x^1 is not null, and it is therefore convenient to introduce the null coordinate v and two space-like coordinates x and y by

$$x^1 = v + \tfrac{1}{2}(x^2 + y^2)FF', \qquad x^2 = xF, \qquad x^3 = yF$$

where $F = F(u)$ satisfies $F'' = -hF$. This now reduces the metric to Rosen form:

II: $$ds^2 = 2\,du\,dv - F^2(u)(dx^2 + dy^2).$$

The metric in region I may be conveniently given by

I: $$ds^2 = 2\,du\,dv - dx^2 - dy^2$$

and taking the field in region III to be of the same form as that in II it is convenient to consider

III: $$ds^2 = 2\,du\,dv - G^2(v)(dx^2 + dy^2).$$

In order to satisfy the junction conditions it is necessary to impose the conditions $F(0) = 1$, $F'(0) = 0$, $G(0) = 1$, $G'(0) = 0$.

In adopting the above metrics in regions II and III it has been assumed that the neutrino fields have the same constant polarization. This is a severe restriction to the generality of the conclusions of this paper. However, these metrics have the interesting property that they admit either an electromagnetic or a neutrino interpretation. This choice therefore enables the differences between the interactions of null electromagnetic and neutrino fields to be demonstrated, since the metric obtained for the interaction region IV when the Einstein-neutrino equations are imposed differs greatly from the metric obtained when the Einstein–Maxwell equations are imposed. A solution for colliding electromagnetic waves of this type has already been given by Bell and Szekeres (1974) for the initial fields given by $F=\cos au$ (ie $h(u)=a^2$) and $G=\cos bv$. These correspond to constant profile null fields. The following results for colliding neutrino fields may immediately be compared with the results of Bell and Szekeres if this choice of metric in regions II and III is taken. For the moment, however, the greater generality is retained.

4. The field equations in the interaction region

In the interaction region IV it is possible to retain null coordinates u and v aligned with the two fields. Taking the propagation vectors of the two fields to be the null vectors l_μ and n_μ, there exist functions $A(u, v)$ and $B(u, v)$ such that

$$Al_\mu = u_{,\mu}, \qquad Bn_\mu = v_{,\mu}$$

provided that the fields continue to follow twist-free null geodesics. With this assumption the tetrad may be constructed as follows

$$\begin{aligned} l_\mu &= A^{-1}\delta^0_\mu & l^\mu &= (0, B, Y^2, Y^3) \\ n_\mu &= B^{-1}\delta^1_\mu & n^\mu &= (A, 0, X^2, X^3) \\ & & m^\mu &= (0, 0, \xi^2, \xi^3). \end{aligned}$$

Now the metric in regions I, II and III has no dependence on the coordinates x and y and therefore intuitively no such dependence is expected in region IV. Accordingly the assumption is made that the metric components and spin coefficients are functions of u and v only, so that when applied to these quantities

$$\mathrm{D} = B\frac{\partial}{\partial v}, \qquad \Delta = A\frac{\partial}{\partial u}, \qquad \delta = 0.$$

Such an assumption may only be justified if it successfully leads to an exact solution. The commutation relations now give

$$\kappa = \nu = 0, \qquad \rho = \bar\rho, \qquad \mu = \bar\mu, \qquad \beta = \bar\alpha, \qquad \bar\tau = \pi = 2\alpha.$$

It is now possible to use the available tetrad transformations to put $\epsilon = 0$ and $\gamma = 0$, and A and B may now both be taken to be unity. The metric and field equations now become

$$\Delta Y^i - \mathrm{D}X^i = -4(\bar\alpha\bar\xi^i + \alpha\xi^i) \tag{4.1}$$

$$\mathrm{D}\xi^i = \rho\xi^i + \sigma\bar\xi^i \tag{4.2}$$

$$\Delta\xi^i = -\mu\xi^i - \bar\lambda\bar\xi^i \tag{4.3}$$

$$\mathrm{D}\rho = \rho^2 + \sigma\bar\sigma + \Phi_{00} \tag{4.4}$$

$$\mathrm{D}\sigma = 2\rho\sigma + \Psi_0 \tag{4.5}$$

$$\mathrm{D}\alpha = 3\rho\alpha + \bar\sigma\bar\alpha + \Phi_{10} \tag{4.6}$$

$$\mathrm{D}\mu = 2\rho\mu + 4\alpha\bar\alpha + \Phi_{11} \tag{4.7}$$

$$D\lambda = \rho\lambda + \mu\bar{\sigma} + 4\alpha^2 + \Phi_{20} \tag{4.8}$$

$$\Delta\rho = -2\rho\mu - 4\alpha\bar{\alpha} - \Phi_{11} \tag{4.9}$$

$$\Delta\sigma = -\mu\sigma - \rho\bar{\lambda} - 4\bar{\alpha}^2 - \Phi_{02} \tag{4.10}$$

$$\Delta\alpha = -3\mu\alpha - \bar{\lambda}\bar{\alpha} - \Phi_{21} \tag{4.11}$$

$$\Delta\mu = -\mu^2 - \lambda\bar{\lambda} - \Phi_{22} \tag{4.12}$$

$$\Delta\lambda = -2\mu\lambda - \Psi_4 \tag{4.13}$$

$$\Psi_1 = 2\rho\bar{\alpha} - 2\sigma\alpha + \Phi_{01}$$

$$\Psi_2 = \rho\mu - \sigma\lambda + \Phi_{11}$$

$$\Psi_3 = 2\mu\alpha - 2\lambda\bar{\alpha} + \Phi_{21}$$

$$\Psi_2 + \Phi_{11} + 12\alpha\bar{\alpha} = 0.$$

Szekeres' approach to the problem of colliding gravitational and electromagnetic waves was to notice that $\alpha = 0$ in regions I, II and III and since in these cases Φ_{10} and Φ_{21} are zero, α must be zero everywhere. He was then able to use the available transformations to put X^i and Y^i zero. However, for colliding neutrino fields Φ_{10} and Φ_{21} are not necessarily zero and the assumption that they are does not lead to an exact solution. It is therefore necessary to consider an alternative approach and the tetrad freedom is used to put $\epsilon = 0$ and $\gamma = 0$.

5. An exact solution

The solutions for colliding gravitational and electromagnetic waves have non-zero contraction and shear in the interaction region. However, for colliding neutrinos there appears to be no reason why shear terms σ and λ should appear if Ψ_0 and Ψ_4 remain zero. The assumption is therefore made that

$$\sigma = 0, \qquad \lambda = 0.$$

This assumption is only finally justified on the grounds that it leads to an exact solution. It is now possible to put

$$\xi^2 = G(u, v)/\sqrt{2}, \qquad \xi^3 = \mathrm{i}G(u, v)/\sqrt{2}$$

and the neutrino conditions become

$$D\phi = \rho\phi - \alpha\psi$$

$$\Delta\psi = -\mu\psi + \bar{\alpha}\phi$$

$$\Phi_{00} = \mathrm{i}k(\psi D\bar{\psi} - \bar{\psi}D\psi)$$

$$\Phi_{01} = \tfrac{1}{2}\mathrm{i}k(\bar{\phi}D\psi - 2\rho\psi\bar{\phi} - \bar{\alpha}\psi\bar{\psi})$$

$$\Phi_{02} = -2\mathrm{i}k\bar{\alpha}\psi\bar{\phi}$$

$$\Phi_{11} = \mathrm{i}k(\bar{\alpha}\phi\bar{\psi} - \alpha\psi\bar{\phi})$$

$$\Phi_{12} = \tfrac{1}{2}\mathrm{i}k(-\psi\Delta\bar{\phi} - 2\mu\psi\bar{\phi} - \bar{\alpha}\phi\bar{\phi})$$

$$\Phi_{22} = \mathrm{i}k(\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi).$$

Equations (4.8, 10) now imply that

$$\alpha = -\tfrac{1}{2}\mathrm{i}k\phi\bar{\psi}$$

so that $\Phi_{11} = -4\alpha\bar{\alpha}$ and equations (4.2, 3, 7, 9) give

$$G = (f(u) + g(v))^{-1/2}$$

$$\rho = -\frac{1}{2}\frac{f'}{f+g}, \qquad \mu = \frac{1}{2}\frac{g'}{f+g}$$

where f and g are arbitrary functions. The remaining equations may now be integrated to give

$$\phi = \frac{1}{\sqrt{(2k)}}\left(\frac{f'}{f+g}\right)^{1/2} \exp[\tfrac{1}{4}\mathrm{i}\ln(f+g) - \tfrac{1}{2}\mathrm{i}\ln f']$$

$$\psi = \frac{1}{\sqrt{(2k)}}\left(\frac{g'}{f+g}\right)^{1/2} \exp[\tfrac{1}{4}\mathrm{i}\ln(f+g) - \tfrac{1}{2}\mathrm{i}\ln g'].$$

All equations are now satisfied except (4.1), but it can readily be seen that it is always possible to choose functions X^i and Y^i satisfying this equation and the necessary boundary conditions.

A global solution may now easily be constructed by keeping g constant for $v<0$ and f constant for $u<0$. These constants may each be taken to be $\frac{1}{2}$ to obtain the flat-space metric required in region I. In region II the metric is now a function of u only, $\phi_A = \phi o_A$ and X^i and Y^i are zero. In region III the metric is a function of v only, $\phi_A = \psi \iota_A$ and X^i and Y^i are zero. In region IV the metric is a function of both u and v given by

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu$$

where

$$l_\mu = \delta^0_\mu, \qquad n_\mu = \delta^1_\mu, \qquad m_\mu = (1/2G)(X^2 + \mathrm{i}X^3,\ Y^2 + \mathrm{i}Y^3, -1, -\mathrm{i}).$$

The required boundary conditions are that

$$\text{on } u = 0, (v \geqslant 0): \qquad f = \tfrac{1}{2}, f' = 0, X^i = Y^i = Y^i_{,u} = 0$$

$$\text{on } v = 0, (u \geqslant 0): \qquad g = \tfrac{1}{2}, g' = 0, X^i = Y^i = X^i_{,v} = 0.$$

The solution in this region is that given above and an appropriate choice of X^i and Y^i is

$$X^2 + \mathrm{i}X^3 = 2\sqrt{2}\int_0^v G\bar{\alpha}\,\mathrm{d}v$$

$$Y^2 + \mathrm{i}Y^3 = -2\sqrt{2}\int_0^u G\bar{\alpha}\,\mathrm{d}u$$

where $G = (f+g)^{-1/2}$ and

$$\alpha = -\frac{\mathrm{i}}{4}\frac{(f'g')^{1/2}}{f+g}\exp(\tfrac{1}{2}\mathrm{i}\ln g' - \tfrac{1}{2}\mathrm{i}\ln f').$$

6. Discussion

When the two fields collide $f+g$ is necessarily positive with $f'(0)=0$ and $g'(0)=0$. However, if the energy density of the neutrino fields are positive then f'' and g'' are negative (this may be deduced from § 3 where $F''=-hF$ etc). This causes a focusing of the two fields, the contraction of the fields becoming infinite on the space-like hypersurface $f+g=0$. On this hypersurface the metric is singular and all components of the Ricci tensor Φ_{AB}, Ψ_1, Ψ_2 and Ψ_3 are all unbounded. It is worth noting in passing that colliding neutrino fields with negative energy density cause each other to diverge so that a singularity in the space–time does not appear.

The solution given above may immediately be compared with the Bell–Szekeres solution for colliding electromagnetic waves if the choice is made that $f=\frac{1}{2}\cos 2au$ for $u\geqslant 0$ and $g=\frac{1}{2}\cos 2bv$ for $v\geqslant 0$. This gives the same expression

$$G^{-2}=\cos(au+bv)\cos(au-bv)$$

but in this case the shear is zero and the terms X^i and Y^i appear. The incoming waves now have constant profile. In region II, $\Phi_{22}=a^2$ and in region III, $\Phi_{00}=b^2$. In the interaction region IV

$$\Phi_{00}=b^2\left[1+\left(\frac{\sin 2au}{\cos 2au+\cos 2bv}\right)^2\right]$$

$$\Phi_{22}=a^2\left[1+\left(\frac{\sin 2bv}{\cos 2au+\cos 2bv}\right)^2\right].$$

The remaining terms being easily calculated. In contrast to this, in the solution for colliding electromagnetic waves the components Φ_{00} and Φ_{22} remain constant, $\Phi_{02}=ab$, and the remaining components of the Ricci tensor and the gravitational components Ψ_A do not appear, so that the metric remains conformally flat and the singularity at $au+bv=\frac{1}{2}\pi$ is removable.

References

Bell P and Szekeres P 1974 *Gen. Rel. Gravit.* **5** 275–86
Griffiths J B 1972 *Int. J. Theor. Phys.* **5** 141–50
Griffiths J B and Newing R A 1971 *J. Phys. A: Gen. Phys.* **4** 208–13
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566–78
—— 1963 *J. Math. Phys.* **4** 998
O'Brien S and Synge J L 1952 *Proc. Dublin Inst. Adv. Stud.* A **9**
Penrose R 1966 *Perspectives in Geometry and Relativity*, ed B Hoffmann (Indiana: Indiana UP) chap 27
Robson E H 1973 *Ann. Inst. Henri Poincaré* A **18** 77–88
Szekeres P 1972 *J. Math. Phys.* **13** 286–94
Wainwright J 1971 *J. Math. Phys.* **12** 828–35